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Distance between Two Arbitrary Unperturbed Orbits

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Abstract. The problem of finding critical points of the distance function between two keplerian elliptic orbits (hence finding distance between them in a sense of set theory) is reduced to determination of all real roots of a trigonometric polynomial of degree eight (Kholshevhikov & Vassiliev 1999). A polynomial of smaller degree with such properties does not exist in non-degenerate cases. Here we extend the results to all 9 cases of conic section ordered pairs. Note, that ellipse–hyperbola and hyperbola–ellipse cases are not equivalent as we exclude the variable marking the position on the second curve.

1. Introduction

The problem of finding a distance between two arbitrary Keplerian ellipses E, E' (in a sense of set theory—the minimal distance between two points lying on E, E') emerged four centuries ago together with discovery of Keplerian laws. An optimal solution was found in the paper (Kholshevhikov & Vassiliev 1999), containing also the detailed discussion and all necessary bibliography. The problem is reduced to solving an equation $g(u) = 0$, g being a trigonometric polynomial in one variable of degree 8 sharp, and it cannot be diminished in non-degenerate cases. Here we extend these results to all types of conic sections.

2. Elliptic Orbits

Let E, E' be two confocal elliptic orbits with Keplerian elements a, e, i, Ω, ω for E and the same with a stroke for E'. In terms of eccentric anomaly u the position vector $\mathbf{r}$ on E is

$$\mathbf{r}/a = \mathbf{P}(\cos u - e) + \mathbf{S}\sin u, \tag{1}$$

where $\mathbf{S} = \sqrt{1-e^2}\mathbf{Q}$ and components of the orthogonal unit vectors $\mathbf{P}, \mathbf{Q}$ *for all types of conic sections* are

$$\begin{aligned}
P_x &= \cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega,\\
P_y &= \cos\omega\sin\Omega + \cos i\sin\omega\cos\Omega,\\
P_z &= \sin i\sin\omega,\\
Q_x &= -\sin\omega\cos\Omega - \cos i\cos\omega\sin\Omega,\\
Q_y &= -\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega,\\
Q_z &= \sin i\cos\omega.
\end{aligned}$$

Using (1) one deduces for a normalized squared distance function

$$\rho(u,u') = \frac{|\mathbf{r}-\mathbf{r}'|^2}{2aa'}$$

an expression

$$\begin{aligned}\rho \;=\; & \rho_0 + (PP'e' - \alpha e)\cos u + P'Se'\sin u + \\ & (PP'e - \alpha' e')\cos u' + PS'e\sin u' - PP'\cos u\cos u' - \\ & PS'\cos u\sin u' - P'S\sin u\cos u' - SS'\sin u\sin u' + \\ & (\alpha/4)e^2\cos 2u + (\alpha'/4)e'^2\cos 2u'. \end{aligned} \tag{2}$$

Here

$$\rho_0 = \frac{\alpha+\alpha'}{2} + \frac{\alpha e^2 + \alpha' e'^2}{4} - PP'ee', \tag{3}$$

$$\alpha = \frac{a}{a'}, \qquad \alpha' = \frac{a'}{a}, \tag{4}$$

PP', PS', $P'S$, and SS' are scalar products of corresponding vectors. Function ρ receives its minimal and maximal value at one of the critical points satisfying equations

$$A\,\sin u' + B\,\cos u' = C, \qquad M\,\sin u' + N\,\cos u' = K\,\sin u'\cos u'. \tag{5}$$

Here

$$\begin{aligned} A &= PS'\,\sin u - SS'\,\cos u, \\ B &= PP'\,\sin u - P'S\,\cos u, \\ C &= e'\,B - \alpha e\sin u\,(1-e\cos u), \\ M &= PP'\,\cos u + P'S\,\sin u + \alpha' e' - PP'\,e, \\ N &= PS'\,e - SS'\,\sin u - PS'\,\cos u, \\ K &= \alpha' e'^2 \end{aligned} \tag{6}$$

are trigonometric polynomials in u of degree 0, 1 or 2.

The system (5) can be reduced (Kholshevhikov & Vassiliev 1999) to an equation in one variable

$$g(u) = 0, \tag{7}$$

g being a trigonometric polynomial of degree 8

$$\begin{aligned} g(u) \;=\; & K^2(A^2-C^2)(B^2-C^2) + 2\,KC\left[NA(A^2-C^2) + MB(B^2-C^2)\right] - \\ & (A^2+B^2)\left[N^2(A^2-C^2) + M^2(B^2-C^2) - 2\,NMAB\right]. \end{aligned} \tag{8}$$

After solving (7) we obtain from the first of equations (5)

$$\cos u' = \frac{BC + mA\sqrt{D}}{A^2+B^2}, \qquad \sin u' = \frac{AC - mB\sqrt{D}}{A^2+B^2} \tag{9}$$

with

$$D = A^2 + B^2 - C^2, \qquad m = \pm 1. \tag{10}$$

We extend now the results above to all types of conic sections. Marking elliptic, hyperbolic and parabolic cases in alphabetic order by symbols 1, 2, 3 we obtain 9 cases $\mathcal{K}_{jk}$ and correspondingly 9 functions $g_{jk}(u)$. For example, $\mathcal{K}_{23}$ means that E is a hyperbola and E' is a parabola. Evidently $g_{23} \neq g_{32}$ and the function (8) coincides with g_{11}.

3. Hyperbolic Cases

1. Let begin with hyperbolic-elliptic case $\mathcal{K}_{21}$. Eccentricity of E is greater than 1, a is negative, the eccentric anomaly and $\sqrt{1-e^2}$ have imaginary values. Instead of (1) we have

$$\mathbf{r}/a = \mathbf{P}(\cosh u - e) - \mathbf{S}\sinh u, \tag{11}$$

$u \in (-\infty, \infty)$ being a hyperbolic analogue of eccentric anomaly, $\mathbf{S} = \sqrt{e^2-1}\mathbf{Q}$.

Any expression in old (real for an ellipse) quantities $\mathbf{S}, u$ can be easily rewritten in new ones (real for a hyperbola) using replacement

$$\mathbf{S} \longmapsto i\,\mathbf{S}, \qquad u \longmapsto i\,u, \tag{12}$$

i being the imaginary unit and never being mixed with the inclination. In particular, this replacement converts (1) to (11) and converts (2) to

$$\begin{aligned} \rho \;=\; & \rho_0 + (PP'e' - \alpha e)\cosh u - P'Se'\sinh u + \\ & (PP'e - \alpha' e')\cos u' + PS'e\sin u' - PP'\cosh u\cos u' - \\ & PS'\cosh u \sin u' + P'S\sinh u\cos u' + SS'\sinh u\sin u' + \\ & (\alpha/4)e^2\cosh 2u + (\alpha'/4)e'^2\cos 2u'. \end{aligned} \tag{13}$$

Relations (5) and function (8) are homogeneous with respect to A, B, C and M, N, K. Hence, after the substitution (12) we may multiply A, B, C or M, N, K by any number, not equal to zero. In particular, we may multiply A, B, C by i and change (6) by

$$\begin{aligned} A &= PS'\sinh u - SS'\cosh u, \\ B &= PP'\sinh u - P'S\cosh u, \\ C &= e'B - \alpha e\sinh u\,(1 - e\sinh u), \\ M &= PP'\cosh u - P'S\sinh u + \alpha' e' - PP'e, \\ N &= PS'e + SS'\sinh u - PS'\cosh u, \\ K &= \alpha' e'^2. \end{aligned} \tag{14}$$

Function g_{21} expression in variables (14) coincides with the expression (8) for g_{11}, as well as (9,10) for $\cos u'$, $\sin u'$, D hold true.

2. In the case $\mathcal{K}_{12}$ we need the replacement

$$A \longmapsto i\,A, \qquad N \longmapsto i\,N.$$

New functions $A, \ldots, K$ retain their meaning (6), meanwhile the expressions (2), (5), (8) change

$$\begin{aligned}\rho &= \rho_0 + (PP'e' - \alpha e)\cos u + P'Se'\sin u + \\ &\quad (PP'e - \alpha' e')\cosh u' - PS'e\sinh u' - PP'\cos u\cosh u' + \\ &\quad PS'\cos u \sinh u' - P'S\sin u\cosh u' + SS'\sin u\sinh u' + \\ &\quad (\alpha/4)e^2\cos 2u + (\alpha'/4)e'^2\cosh 2u',\end{aligned} \tag{15}$$

$$-A\,\sinh u' + B\,\cosh u' = C, \quad M\,\sinh u' + N\,\cosh u' = K\,\sinh u'\cosh u', \tag{16}$$

$$\begin{aligned}g_{12}(u) &= -K^2(A^2+C^2)(B^2-C^2) + \\ &\quad 2\,KC\left[NA(A^2+C^2) + MB(B^2-C^2)\right] + \\ &\quad (A^2-B^2)\left[N^2(A^2+C^2) + M^2(B^2-C^2) + 2\,NMAB\right].\end{aligned} \tag{17}$$

Instead of (9), (10) we have

$$\cosh u' = \frac{BC + mA\sqrt{D}}{B^2 - A^2}, \qquad \sinh u' = \frac{AC + mB\sqrt{D}}{B^2 - A^2} \tag{18}$$

with

$$D = A^2 + C^2 - B^2, \qquad m = \pm 1. \tag{19}$$

3. In the case $\mathcal{K}_{22}$ we ought to replace

$$A \longmapsto -A, \qquad B \longmapsto i\,B, \qquad C \longmapsto i\,C, \qquad N \longmapsto i\,N,$$

and for new quantities $A, \ldots, K$ the expressions (14) hold true as in the case $\mathcal{K}_{21}$. On the contrary, the expressions (16)–(19) hold true as in the case $\mathcal{K}_{12}$. Finally,

$$\begin{aligned}\rho &= \rho_0 + (PP'e' - \alpha e)\cosh u - P'Se'\sinh u + \\ &\quad (PP'e - \alpha' e')\cosh u' - PS'e\sinh u' - PP'\cosh u\cosh u' + \\ &\quad PS'\cosh u \sinh u' + P'S\sinh u\cosh u' - SS'\sinh u\sinh u' + \\ &\quad (\alpha/4)e^2\cosh 2u + (\alpha'/4)e'^2\cosh 2u'.\end{aligned} \tag{20}$$

4. Parabolic Cases

The best way to treat a parabolic orbit is to present it as a limiting case ($\varepsilon \to 0$) of an ellipse

$$a = \frac{q}{2\varepsilon^2}, \qquad e = 1 - 2\varepsilon^2, \qquad u = 2\varepsilon\sigma, \tag{21}$$

q, σ being the pericentric distance and the tangent of the half of the true anomaly. The substitution (21) converts (1) to

$$\mathbf{r}/q = \mathbf{P}\left[(1-\sigma^2) + \mathcal{O}(\varepsilon^2)\right] + \mathbf{Q}\left[2\sigma + \mathcal{O}(\varepsilon^2)\right]. \tag{22}$$

1. In the cases $\mathcal{K}_{31}$ and $\mathcal{K}_{32}$ we may deal with

$$\rho_{31} = \rho_{32} = \frac{|\mathbf{r} - \mathbf{r}'|^2}{qa'} = \frac{\rho}{\varepsilon^2},$$

$$\beta = \frac{q}{a'} = 2\varepsilon^2\alpha, \qquad \beta' = \frac{a'}{q} = \frac{\alpha'}{2\varepsilon^2}, \qquad \mathbf{S} = 2\varepsilon\mathbf{Q}\left[1 + \mathcal{O}(\varepsilon^2)\right].$$

Taking (21) into account we have after passing to the limit ($\varepsilon \to 0$)

$$\begin{aligned} \rho_{31} &= \beta\left(1+\sigma^2\right)^2 + \beta'\left(1 + \frac{1}{2}e'^2\right) + 2PP'e'\left(1-\sigma^2\right) + \\ &\quad 4\sigma e'P'Q - 2\left[(1-\sigma^2)PP' + 2\sigma P'Q + \beta'e'\right]\cos u' - \\ &\quad 2\left[(1-\sigma^2)PS' + 2\sigma QS'\right]\sin u' + \frac{1}{2}\beta'e'^2\cos 2u'. \end{aligned} \tag{23}$$

The system (5) holds true if we devide A, B, C by 2ε and M, N, K by $2\varepsilon^2$ and then pass to the limit. So we have for $\mathcal{K}_{31}$

$$\begin{aligned} A &= PS'\sigma - QS', \\ B &= PP'\sigma - P'Q, \\ C &= e'B - \beta\sigma(1+\sigma^2), \\ M &= PP'(1-\sigma^2) + 2\sigma P'Q + \beta'e', \\ N &= -PS'(1-\sigma^2) - 2\sigma QS', \\ K &= \beta'e'^2. \end{aligned} \tag{24}$$

Due to the homogeneity of g we may use the expression (8) for g_{31}, as well as (9), (10) for u'.

2. The case $\mathcal{K}_{32}$ may be deduced from $\mathcal{K}_{12}$ by the similar procedure. So

$$\begin{aligned} \rho_{32} &= \beta\left(1+\sigma^2\right)^2 + \beta'\left(1 + \frac{1}{2}e'^2\right) + 2PP'e'\left(1-\sigma^2\right) + \\ &\quad 4\sigma e'P'Q - 2\left[(1-\sigma^2)PP' + 2\sigma P'Q + \beta'e'\right]\cosh u' + \\ &\quad 2\left[(1-\sigma^2)PS' + 2\sigma QS'\right]\sinh u' + \frac{1}{2}\beta'e'^2\cosh 2u'. \end{aligned} \tag{25}$$

Relations (24) are valid and we may use the expression (17) for g_{32}, as well as (18), (19) for u'.

3. Cases $\mathcal{K}_{13}$ and $\mathcal{K}_{23}$ are more complicated and we omit them.

4. In the case $\mathcal{K}_{33}$ we introduce

$$\rho_{33} = \frac{|\mathbf{r} - \mathbf{r}'|^2}{qq'} = \frac{\rho_{31}}{2\varepsilon^2},$$

$$\gamma = \frac{q}{q'} = 2\frac{\beta}{2\varepsilon^2}, \qquad \gamma' = \frac{q'}{q} = 2\varepsilon^2\beta', \qquad \mathbf{S}' = 2\varepsilon\mathbf{Q}\left[1 + \mathcal{O}(\varepsilon^2)\right].$$

After passing to the limit

$$\begin{aligned}\rho_{33} &= \gamma\left(1+\sigma^2\right)^2+\gamma'\left(1+\sigma'^2\right)^2-2PP'\left(1-\sigma^2\right)\left(1-\sigma'^2\right)- \\ &\quad 4\sigma\left(1-\sigma'^2\right)P'Q-4\left(1-\sigma^2\right)^2\sigma'PQ'-8\sigma\sigma'QQ'. \qquad (26)\end{aligned}$$

One uses (24) for calculations of $A,\ldots,K$. Relations

$$B\sigma'^2-2A\sigma'+C=0, \qquad K\sigma'^3+M\sigma'+N=0,$$

and

$$\begin{aligned}g_{33} &= K^2\,C^3+2\,K\left[NA(4A^2-3BC)+MC(2A^2-BC)\right]+ \\ &\quad B^2\left(N^2B+M^2C+2MNA\right)\end{aligned}$$

are taken to find σ, σ'.

5. Conclusions

We have proposed 9 functions g_{jk} solving the problem of finding distance between E and E' in all possible combinations of conic sections. In the non-parabolic cases ($j \le 2$, $k \le 2$) functions g_{jk} are trigonometric or hyperbolic polynomials of degree 8. Remember that the degree of a corresponding algebraic polynomial must be multiplied by a factor 2. In the cases when only one of the orbits is parabolic functions g_{31}, g_{32}, g_{13}, g_{23} are trigonometric or hyperbolic polynomials of degree 6 or algebraic polynomial of degree 12. Hence, all of them can be reduced to an algebraic polynomial of degree 12. In the case when the both orbits are parabolic, function g_{33} represents an algebraic polynomial of degree 9 only.

For non-diagonal elements of $\mathcal{K}_{jk}$ it is useful to choose the simplest function between g_{jk} and g_{kj}. Supposing a trigonometric equation being simpler than a hyperbolic one, we recommend the function g_{12} for the case ellipse–hyperbola. Supposing an algebraic equation being simpler than a trigonometric or hyperbolic one, we recommend the functions g_{31} and g_{32} for the cases parabola–ellipse and parabola–hyperbola.

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References

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