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Collinear Lagrange Point Solutions in the Circular Restricted Three-Body Problem with Radiation Pressure using Fortran

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Abstract. The effects of radiation pressure in the circular restricted three-body problem displace the Lagrangian equilibrium points from their classical locations. Components of Newton-Raphson solutions to displaced collinear Lagrange points are presented here. Using Fortran 95, solution convergence for valid approximate zeros (0.07 for r_2 at L_1, L_2 and 1 for r_1 at L_3) and radiation pressure relevant to asteroidal dust particles ($\beta = 0.2$), was explored quantitatively.

1. Equations of Motion in the Circular Restricted Three-Body Problem with Radiation Pressure

The collinear Lagrange points L_1, L_2 and L_3 are equilibrium solutions of the three-body problem, lying along a line joining the two primaries of the system, m_1 and m_2. Comprehensive three-body system dynamics summaries are given by Szebehely (1967) and Marchal (1990).

The synodic equations of motion of the infinitesimal body, m_3, of the circular restricted three-body problem, CRTBP, are given by (Murray and Dermott 1999)

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{dx},\ \ddot{y} + 2n\dot{x} = \frac{\partial U}{dy},\ \ddot{z} = \frac{\partial U}{dz} \tag{1}$$

where n is the common angular velocity with which the primaries orbit their centre of mass. It is assumed the primaries move in the (x, y) plane of a Cartesian (x, y, z) coordinate system centred on their centre of mass, with $+x$ in the direction of m_2. Here U is a pseudo-potential of the CRTBP (Liou et al. 1995, Murray and Dermott 1999),

$$U \equiv \frac{n^2}{2}(x^2 + y^2) + \frac{\mu_1(1-\beta)}{r_1} + \frac{\mu_2}{r_2}. \tag{2}$$

A normalised system of dimensionless units giving $m_1 + m_2 = 1$, $\mathrm{d}\overline{m_1 m_2} = 1$ and $G = 1$, where G is the universal gravitational constant, was adopted here. In Eq. 2, $\mu_1 = m_1/(m_1+m_2)$, $\mu_2 = m_2/(m_1+m_2)$, $r_1^2 = (x+\mu_2)^2+y^2+z^2$, $r_2^2 = (x-\mu_1)^2 + y^2 + z^2$ and β is the ratio of solar radiation pressure to gravitational force experienced by m_3. β is of order unity for asteroidal dust particles of radius 0.1 μm to 1 μm (Schuerman 1980) in planetary Lagrange points, for example.

2. Locations of the Collinear Lagrange Points in the Circular Restricted Three-Body Problem with Radiation Pressure

Adopting a normalised mean motion of $n = 1$, U may be rewritten as Eq. 3, below.

$$U = \mu_1 \left(\frac{r_1^2}{2} + \frac{1-\beta}{r_1} \right) + \mu_2 \left(\frac{r_2^2}{2} + \frac{1}{r_2} \right) - \frac{\mu_1 \mu_2}{2} \tag{3}$$

Synodic acceleration and velocity of m_3 at rest in a Lagrange point are both 0. For U as given in Eq. 3, which is a function of (r_1, r_2), rather than explicitly of (x, y), this can be stated as,

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x} = 0, \ \frac{\partial U}{\partial y} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y} = 0. \tag{4}$$

From Eqs. 3. and 4., the system Lagrange points must satisfy Eq. 5, below.

$$\begin{aligned} \mu_1 \left(r_1 - \frac{1-\beta}{r_1^2} \right) \frac{x+\mu_2}{r_1} + \mu_2 \left(r_2 - \frac{1}{r_2^2} \right) \frac{x-\mu_1}{r_2} &= 0, \\ \mu_1 \left(r_1 - \frac{1-\beta}{r_1^2} \right) \frac{y}{r_1} + \mu_2 \left(r_2 - \frac{1}{r_2^2} \right) \frac{y}{r_2} &= 0. \end{aligned} \tag{5}$$

At L_1, $y = 0$, $r_1 + r_2 = 1$, $r_1 = x + \mu_2$, $r_2 = \mu_1 - x$, at L_2, $y = 0$, $r_1 - r_2 = 1$, $r_1 = x + \mu_2$, $r_2 = x - \mu_1$ and at L_3, $y = 0$, $r_2 - r_1 = 1$, $r_1 = -x - \mu_2$ and $r_2 = x - \mu_1$. Substituting for L_1, L_2 and L_3 respectively into the first component of Eq. 5 and expressing as mass ratios gives,

$$at L_1, \quad \frac{m_2}{m_1} = r_2^2 \frac{(1-r_2)^3 + \beta - 1}{(r_2^3 - 1)(1-r_2)^2}, \tag{6}$$

$$at L_2, \quad \frac{m_2}{m_1} = r_2^2 \frac{(r_2^3 + 3r_2^2 + 3r_2 + \beta)}{(1+r_2)^2(1-r_2^3)}, \tag{7}$$

$$at L_3, \quad \frac{m_2}{m_1} = \frac{(1-\beta-r_1^3)(1+r_1)^2}{r_1^3(r_1^2 + 3r_1 + 3)}. \tag{8}$$

The Newton-Raphson algorithm $r_{2i+1} = r_{2i} - f(r_{2i})/f'(r_{2i})$ may be used to solve Eqs. 6 to 8 and determine r_2 for L_1, L_2 and r_1 for L_3. Newton-Raphson components for r_2 at L_1 are given by Eqs. 9 and 10, r_2 at L_2 by Eqs. 11 and 12 and r_1 at L_3 by Eqs. 13 and 14.

$$f(r_2) = \frac{r_2^5 - 3r_2^4 + 3r_2^3 - \beta r_2^2}{-r_2^5 + 2r_2^4 - r_2^3 + r_2^2 - 2r_2 + 1} - \frac{m_2}{m_1}. \tag{9}$$

$$\begin{aligned} f'(r_2) = {} & \frac{-r_2^8 + 4r_2^7 - 3\beta r_2^6 - (7-2\beta)2r_2^5 + (26-\beta)r_2^4 - 24r_2^3}{r_2^{10} - 4r_2^9 + 6r_2^8 - 6r_2^7 + 9r_2^6 - 12r_2^5 + 9r_2^4 - 6r_2^3 + 6r_2^2 - 4r_2 + 1} \\ & \frac{+(9-2\beta)r_2^2 - 2\beta r_2}{r_2^{10} - 4r_2^9 + 6r_2^8 - 6r_2^7 + 9r_2^6 - 12r_2^5 + 9r_2^4 - 6r_2^3 + 6r_2^2 - 4r_2 + 1}. \end{aligned} \tag{10}$$

$$f(r_2) = \frac{r_2^5 + 3r_2^4 + 3r_2^3 + \beta r_2^2}{-r_2^5 - 2r_2^4 - r_2^3 + r_2^2 + 2r_2 + 1} - \frac{m_2}{m_1}. \quad (11)$$

$$f'(r_2) = \frac{r_2^8 + 4r_2^7 + (6+3\beta)r_2^6 + (14+4\beta)r_2^5 + (26+\beta)r_2^4 + 24r_2^3}{r_2^{10} + 4r_2^9 + 6r_2^8 + 2r_2^7 - 7r_2^6 - 12r_2^5 - 7r_2^4 + 2r_2^3 + 6r_2^2 + 4r_2 + 1} \frac{+(9+2\beta)r_2^2 + 2\beta r_2}{r_2^{10} + 4r_2^9 + 6r_2^8 + 2r_2^7 - 7r_2^6 - 12r_2^5 - 7r_2^4 + 2r_2^3 + 6r_2^2 + 4r_2 + 1}. \quad (12)$$

$$f(r_1) = \frac{-r_1^5 - 2r_1^4 - r_1^3 + (1-\beta)r_1^2 + (1-\beta)2r_1 - \beta + 1}{r_1^5 + 3r_1^4 - 3r_1^3} - \frac{m_2}{m_1}. \quad (13)$$

$$f'(r_1) = \frac{-r_1^6 - 4r_1^5 - (2-\beta)3r_1^4 - (1+\beta)14r_1^3 - (1-\beta)(26r_1^2 + 24r_1 + 9)}{r_1^8 + 6r_1^7 + 15r_1^6 + 18r_1^5 + 9r_1^4}. \quad (14)$$

3. Newton-Raphson Solution Convergence

The Newton-Raphson algorithm provides rapid solution convergence, given a sensible initial solution approximation (Fitz-Gerald and Peckham 1996). Solution convergence in the CRTBP with radiation pressure is explored here using a Sun-Jupiter-particle system as a test case ($m_1 \approx 0.99905$, $m_2 \approx 0.00095$ (Murray and Dermott 1999)). In the classical $\beta = 0$ case, this system gives r_2 for $L_1 \approx 0.066674$, r_2 for $L_2 \approx 0.069777$ and r_1 for $L_3 \approx 0.99944$. $r_2 = 0.07$ for L_1, L_2 and $r_1 = 1$ for L_3 were thus used as initial solution approximations.

For an asteroidal dust particle experiencing $\beta = 0.2$, Newton-Raphson solution convergence on r_2 for L_1, r_2 for L_2 and r_1 for L_3 are presented in Tables 1 to 3, below. All computations were performed in double precision arithmetic using Fortran 95 with an open source g95 compiler. Compared with L_1, the initial L_2 solution approximation was closer to the final solution and converges correspondingly faster. An increase in r_2 for L_1 and decrease in r_1 for L_3 and r_2 for L_2, compared with the $\beta = 0$ result, is consistent with Liou et al. (1995).

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Table 1. Newton-Raphson convergence on r_2 for L_1, $\beta = 0.2$

i	r_{2i}	$f(r_2)$	$f'(r_2)$
0	0.07	-9.7919×10^{-4}	8.7022×10^{-3}
1	0.18252	1.1770×10^{-2}	2.1199×10^{-1}
2	0.12700	1.9011×10^{-3}	7.7830×10^{-2}
3	0.10257	5.5556×10^{-5}	4.0570×10^{-2}
4	0.10120	-1.6324×10^{-5}	3.8834×10^{-2}
5	0.10162	5.4269×10^{-6}	3.9363×10^{-2}
6	0.10149	-1.7375×10^{-6}	3.9189×10^{-2}
7	0.10153	5.6319×10^{-7}	3.9245×10^{-2}
8	**0.10152**	-1.8183×10^{-7}	3.9227×10^{-2}
9	**0.10152**	5.8780×10^{-8}	3.9233×10^{-2}
10	**0.10152**	-1.8994×10^{-8}	$\mathbf{3.9231 \times 10^{-2}}$
11	**0.10152**	6.1386×10^{-9}	$\mathbf{3.9231 \times 10^{-2}}$

Table 2. Newton-Raphson convergence on r_2 for L_2, $\beta = 0.2$

i	r_{2i}	$f(r_2)$	$f'(r_2)$
0	0.07	8.6525×10^{-4}	6.3323×10^{-2}
1	0.056336	1.2284×10^{-4}	4.5740×10^{-2}
2	0.053650	4.2824×10^{-6}	4.2567×10^{-2}
3	**0.053550**	5.8896×10^{-9}	$\mathbf{4.2449 \times 10^{-2}}$
4	**0.053550**	1.1069×10^{-14}	$\mathbf{4.2449 \times 10^{-2}}$

Table 3. Newton-Raphson convergence on r_1 for L_3, $\beta = 0.2$

i	r_{1i}	$f(r_1)$	$f'(r_1)$
0	1	-1.1524×10^{-1}	-1.5184
1	0.92410	6.7260×10^{-3}	-1.9744
2	0.92751	5.0955×10^{-4}	-1.9501
3	0.92777	3.5980×10^{-5}	-1.9483
4	**0.92779**	2.5263×10^{-6}	**-1.9482**
5	**0.92779**	1.7731×10^{-7}	**-1.9482**